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LETTER TO THE EDITOR

Elimination of turbulence modes using a conditional average with asymptotic freedom

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Abstract. The part of the nonlinear term in the Navier–Stokes equation which represents coupling to the small-scale modes may be averaged out by introducing a weak conditional average with asymptotic freedom in wavenumber. A residual deterministic part, while important for individual realizations, makes a negligible contribution to the renormalization of the dissipation rate. This is because the full ensemble average, needed to establish the energy balance, relaxes the constraint on the conditional average.

The application of renormalization group methods to dynamical problems in microscopic physics requires an average over small scales in which large scales are held fixed [1]. Unfortunately, the corresponding procedure for classical nonlinear systems, such as Navier–Stokes turbulence, is impossible, *in principle*, because of the deterministic nature of such systems. Recently, it has been proposed that the chaotic nature of turbulence may justify the use of an approximate conditional average [2]. In this paper we argue that the conditional elimination of a band of high-wavenumber modes may be accomplished in terms of a deterministic part, which has a coherent phase relation with the retained modes, and a random part, which is asymptotically free and may be averaged out with the introduction of an effective viscosity. The reduction of the number of modes takes place at a constant rate of energy dissipation, and it is further argued that the renormalization of this quantity can be adequately represented by the incoherent part only. This is because the full ensemble average, needed for the spectral energy balance, tends to 'lift' the constraint on the conditional average.

We consider incompressible fluid turbulence, as governed by the solenoidal Navier–Stokes equation (NSE)

$$(\partial_t + \nu_0 k^2) u_\alpha(\boldsymbol{k}, t) = M_{\alpha\beta\gamma}(\boldsymbol{k}) \int \mathrm{d}^3 j\, u_\beta(\boldsymbol{j}, t) u_\gamma(\boldsymbol{k}-\boldsymbol{j}, t) \tag{1}$$

where ν_0 is the kinematic viscosity of the fluid,

$$M_{\alpha\beta\gamma}(\boldsymbol{k}) = (2\mathrm{i})^{-1}[k_\beta D_{\alpha\gamma}(\boldsymbol{k}) + k_\gamma D_{\alpha\beta}(\boldsymbol{k})] \tag{2}$$

and the projector $D_{\alpha\beta}(\boldsymbol{k})$ is expressed in terms of the Kronecker delta $\delta_{\alpha\beta}$ as

$$D_{\alpha\beta}(\boldsymbol{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta |\boldsymbol{k}|^{-2}. \tag{3}$$

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In order to pose a specific problem, we restrict our attention to stationary, isotropic, homogeneous turbulence, with dissipation rate ε and zero mean velocity. We also introduce an upper cutoff wavenumber $K_{\max}$, which is defined through the dissipation integral

$$\varepsilon = \int_0^\infty 2\nu_0 k^2 E(k)\,\mathrm{d}k \simeq \int_0^{K_{\max}} 2\nu_0 k^2 E(k)\,\mathrm{d}k \tag{4}$$

where $E(k)$ is the energy spectrum, so ensuring that $K_{\max}$ is of the same order of magnitude as the Kolmogorov dissipation wavenumber.

We then filter the velocity field at $|\boldsymbol{k}| \equiv k = K_c$, where $0 < K_c < K_{\max}$, according to

$$u_\alpha(\boldsymbol{k}, t) = \begin{cases} u_\alpha^-(\boldsymbol{k}, t) & \text{for} \quad 0 < k < K_c \\ u_\alpha^+(\boldsymbol{k}, t) & \text{for} \quad K_c < k < K_{\max}. \end{cases} \tag{5}$$

The NSE may be decomposed using (5), to give

$$(\partial_t + \nu_0 k^2)u_k^- = M_k^-(u_j^- u_{k-j}^- + 2u_j^- u_{k-j}^+ + u_j^+ u_{k-j}^+) \tag{6}$$

$$(\partial_t + \nu_0 k^2)u_k^+ = M_k^+(u_j^- u_{k-j}^- + 2u_j^- u_{k-j}^+ + u_j^+ u_{k-j}^+) \tag{7}$$

where, for simplicity, all vector indices and independent variables are contracted into a single subscript.

In order to obtain an expression for the average effect of the high-wavenumber modes upon a particular low-wavenumber mode, we need to average out the u^+ whilst holding the u^- constant. This requires a *conditional average* $\langle\cdot\rangle_c$, such that

$$\langle u_\alpha^-(\boldsymbol{k}, t)\rangle_c = u_\alpha^-(\boldsymbol{k}, t). \tag{8}$$

This is the *only* rigorous property we can attribute to the conditional average, and it should also be noted that it is vital to distinguish between this operation and that of a filtered ensemble average.

To establish the statistical properties of $u_\alpha(\boldsymbol{k}, t)$ we consider an ensemble $\mathcal{W}$ consisting of the set of M time-independent realizations $\{W_\alpha^{(i)}(\boldsymbol{k})\}$, each realization† being labelled by an integer i. Subject to certain weak conditions, the ensemble average is

$$\langle u_\alpha(\boldsymbol{k}, t)\rangle = \lim_{M\to\infty} \frac{1}{M} \sum_{i=1}^{M} W_\alpha^{(i)}(\boldsymbol{k}) = \bar{U}_\alpha(\boldsymbol{k}) \tag{9}$$

where $\bar{U}_\alpha(\boldsymbol{k})$ is the time average of $u_\alpha(\boldsymbol{k}, t)$. This procedure can then be extended to any well behaved functional, $F[u_\alpha(\boldsymbol{k}, t)]$, thus:

$$\langle F[u_\alpha(\boldsymbol{k}, t)]\rangle = \lim_{M\to\infty} \frac{1}{M} \sum_{i=1}^{M} F[W_\alpha^{(i)}(\boldsymbol{k})]. \tag{10}$$

Now we consider how to perform a *conditional* average. To do this, we first select a subensemble, $\mathcal{Y} \equiv \left\{Y_\alpha^{(i)}(\boldsymbol{k})\right\} \subset \mathcal{W}$, and choose the members of this *biased* subensemble to be those N $(N \leqslant M)$ members of $\mathcal{W}$ satisfying the criterion

$$\lim_{\delta\to 0}(\max|\theta^-(k)W_\alpha^{(i)}(\boldsymbol{k}) - u_\alpha^-(\boldsymbol{k}, t_1)| \leqslant \delta) \tag{11}$$

where t_1 is some fixed time and $\theta^-(k) = 1$ for $0 < k < K_c$, and zero otherwise. The conditional average is then obtained by generalizing (9) and (10) to the biased subensemble, namely,

$$\langle u_\alpha(\boldsymbol{k}, t)\rangle_c = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} Y_\alpha^{(i)}(\boldsymbol{k}) \tag{12}$$

† Note that this differs from the formulation in [2], where each realization was time-dependent.

and

$$\langle F[u_\alpha(\boldsymbol{k},t)]\rangle_c = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} F[Y_\alpha^{(i)}(\boldsymbol{k})]. \tag{13}$$

It follows by construction that (8) holds, since from (11) and (12)

$$\langle u_\alpha^-(\boldsymbol{k},t)\rangle_c = \lim_{N\to\infty}\frac{1}{N}[Nu_\alpha^-(\boldsymbol{k},t)] = u_\alpha^-(\boldsymbol{k},t). \tag{14}$$

The difficulty now facing us lies in the nature of the subensemble, which is an example of deterministic chaos. This can be seen if we consider two extreme scenarios for the behaviour of u^+ under the conditional average. Firstly, if we assume that the subensemble is *strictly* deterministic, then in this instance u_k^+ is fully determined by prescribing u_k^-. Accordingly, equation (8) implies that $\langle u_j^- u_{k-j}^-\rangle_c = u_j^- u_{k-j}^-$, $\langle u_j^- u_{k-j}^+\rangle_c = u_j^- u_{k-j}^+$ and $\langle u_j^+ u_{k-j}^+\rangle_c = u_j^+ u_{k-j}^+$. Thus, the low-pass filtered NSE, equation (6), reduces back to itself under the conditional average. Secondly, if we assume that the subensemble is purely random, it follows that in this case, u_k^+ is independent of u_k^-. Hence, applying the conditional average to the low-pass filtered NSE, we find

$$(\partial_t + \nu_0 k^2)u_k^- = M_k^- u_j^- u_{k-j}^-$$

the $u_j^- u_{k-j}^+$ term being zero since the ensemble average of u^+ is zero, whilst the $u_j^+ u_{k-j}^+$ term is zero due to homogeneity. Thus in this scenario it appears that there is *no* effect of nonlinear coupling.

In reality we are faced with a situation somewhere between these two extremes, and so we replace our criterion for members of the biased subensemble, equation (11), which is equivalent to the first of these situations if $\delta = 0$, by the less precise criterion

$$\max|\theta^-(k)W_\alpha^{(i)}(\boldsymbol{k}) - u_\alpha^-(\boldsymbol{k},t_1)| \leqslant \xi \tag{15}$$

where, in general, ξ is of the order of the turbulent velocities involved.

To obtain a non-trivial conditional average we must now identify those circumstances in which ξ may be neglected as being, in some sense, small. A measure of the 'smallness of ξ' can be identified by constructing the subensemble as

$$W_\alpha^{(i)}(\boldsymbol{k}) = u_\alpha^-(\boldsymbol{k},t_1) + \phi_\alpha^{(i)}(\boldsymbol{k},t_1) \tag{16}$$

where i is any label satisfying (15). If we then further restrict the subensemble to be such that the set $\{\phi_\alpha^{(i)}(\boldsymbol{k},t_1)\}$ satisfies (8), we find that

$$\langle u_j^- u_{k-j}^-\rangle_c = u_j^- u_{k-j}^- + \langle\phi_j\phi_{k-j}\rangle_c. \tag{17}$$

Thus in order to maintain form invariance of the NSE under conditional averaging, we require

$$\langle\phi_j\phi_{k-j}\rangle_c \to 0 \tag{18}$$

in some limit. This is our criterion for the smallness of ξ.

If we further suppose that chaos and unpredictability are local characteristics of turbulence, and there is support for such a view [3, 4], then if K_c and $K_{\max}$ are sufficiently far apart we might expect, due to the development of unpredictability as k is increased above K_c, that the effect of the constraint given in equation (15) would die away, such that

$$\lim_{k\to K_{\max}}\langle u_\alpha^+(\boldsymbol{k},t)\rangle_c \to \langle u_\alpha^+(\boldsymbol{K}_{\max},t)\rangle. \tag{19}$$

We refer to this property as *asymptotic freedom*. In order to extend this concept to higher-order moments, we introduce the following *hypothesis of local chaos*:

'For sufficiently large Reynolds' number and corresponding $K_{\max}$, there exists a cut-off wavenumber $K_c < K_{\max}$, such that a mixed conditional moment involving p low-wavenumber and r high-wavenumber modes takes the limiting form:

$$\lim_{\xi\to 0}\langle u_\alpha^-(\boldsymbol{k}_1,t)u_\beta^-(\boldsymbol{k}_2,t)\ldots u_\gamma^-(\boldsymbol{k}_p,t)u_\delta^+(\boldsymbol{k}_{p+1},t)u_\epsilon^+(\boldsymbol{k}_{p+2},t)\ldots u_\sigma^+(\boldsymbol{k}_{p+r},t)\rangle_c$$
$$\to u_\alpha^-(\boldsymbol{k}_1,t)u_\beta^-(\boldsymbol{k}_2,t)\ldots u_\gamma^-(\boldsymbol{k}_p,t) \times \lim_{\{\cdot\}\to K_{\max}}\langle u_\delta^+(\boldsymbol{k}_{p+1},t)u_\epsilon^+(\boldsymbol{k}_{p+2},t)\ldots u_\sigma^+(\boldsymbol{k}_{p+r},t)\rangle \qquad (20)$$

where $\lim_{\{\cdot\}\to K_{\max}}$ means take the limit for all wavevector arguments of the u^+ modes, with the condition of equation (18) satisfied as a corollary'.

This provides our definition of an asymptotic conditional average and we should emphasize that the numerical simulations of Machiels [4] provide independent verification of this behaviour. It may be used to evaluate all terms involving mixed products of u^- with u^+. For example,

$$\lim_{\xi\to 0}\langle u_j^- u_{k-j}^+\rangle_c = u_j^- \lim_{\{\cdot\}\to K_{\max}}\langle u_{k-j}^+\rangle = 0 \qquad (21)$$

since $\langle u_\alpha^+(\boldsymbol{k},t)\rangle = 0$. Note also, that the hypothesis as stated is more general than is necessary, since we shall only need to consider products containing at most two u^- modes.

If we then take the conditional average of the low-pass filtered NSE, equation (6), we obtain

$$(\partial_t + \nu_0 k^2)u_k^- = M_k^-\{\langle u_j^- u_{k-j}^-\rangle_c + 2\langle u_j^- u_{k-j}^+\rangle_c + \langle u_j^+ u_{k-j}^+\rangle_c\} \qquad (22)$$

where the conditional average of u_k^- on the left-hand side has been evaluated using (8). This equation may be further rewritten as

$$(\partial_t + \nu_0 k^2)u_k^- = M_k^- u_j^- u_{k-j}^- + S^-(k|K_c) + M_k^- \lim_{\xi\to 0}\langle u_j^+ u_{k-j}^+\rangle_c \qquad (23)$$

where

$$S^-(k|K_c) = M_k^-\left\{\langle \phi_j^- \phi_{k-j}^-\rangle_c + 2\langle u_j^- u_{k-j}^+\rangle_c + \langle u_j^+ u_{k-j}^+\rangle_c - \lim_{\xi\to 0}\langle u_j^+ u_{k-j}^+\rangle_c\right\}. \qquad (24)$$

It should also be noted that the hypothesis *must* hold for $K_c \to 0$, as in this instance equation (22) reduces to the Reynolds equation, with $u_\alpha(\boldsymbol{k},t) \to \bar{U}_\alpha(\boldsymbol{k})$ as given by (9).

Our hypothesis does not explicitly tell us how evaluate the conditional average in (23), which involves a non-trivial projection of a product of u^+ modes in the Hilbert space of the u^- modes, but we may use the high-pass filtered NSE, equation (7), to obtain a governing equation for this quantity. To do this, we use (7) to write equations for u_j^+ and u_{k-j}^+, multiply these equations by u_{k-j}^+ and u_j^+, respectively, add the resulting equations together, and then take the conditional average. After some rearrangement of dummy variables, this gives

$$\lim_{\xi\to 0}(\partial_t + \nu_0 j^2 + \nu_0|\boldsymbol{k}-\boldsymbol{j}|^2)\langle u_j^+ u_{k-j}^+\rangle_c = \lim_{\xi\to 0} 2M_j^+ \times\{\langle u_p^- u_{j-p}^- u_{k-j}^+\rangle_c + 2\langle u_p^- u_{j-p}^+ u_{k-j}^+\rangle_c + \langle u_p^+ u_{j-p}^+ u_{k-j}^+\rangle_c\}. \qquad (25)$$

Applying the hypothesis as given by equation (20), it is easily seen that the first term on the right-hand side of (25) is zero, since in the limit it involves the ensemble average of u_k^+, while the second term gives rise to a term linear in u_k^-. The third term may be evaluated by iterating the above procedure to form a dynamical equation for $\langle u_p^+ u_{j-p}^+ u_{k-j}^+\rangle_c$, which in turn gives rise to higher-order moments.

In general, we can show that a similar pattern occurs for all higher-order moments involving only products of u^+. That is, each such moment gives rise to a moment involving two u^- modes, which in general, has to be zero for consistency in its wavevector arguments, a term linear in

u_k^-, and a moment involving only u^+ modes of next higher order. Hence we may write the general result

$$M^-_{\alpha\beta\gamma}(\boldsymbol{k})\int \mathrm{d}^3 j\ \lim_{\xi\to 0}\langle u^+_\beta(\boldsymbol{j},t)u^+_\gamma(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c = \int_{-\infty}^{t}\mathrm{d}s\ A(\boldsymbol{k},t-s)u^-_\alpha(\boldsymbol{k},s) \tag{26}$$

where $A(\boldsymbol{k},t-s)$ has the form

$$\begin{aligned} A(\boldsymbol{k},t-s) = \int \mathrm{d}^3 j\ &\exp[-(\nu_0 j^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2)(t-s)] \\ &\times\Big\{4M^-_k M^+_j \lim_{\{\cdot\}\to K_{\max}}\langle u^+_{j-p}u^+_{k-j}\rangle \\ &+24M^-_k M^+_j L^{-1}_{03} M^+_p \lim_{\{\cdot\}\to K_{\max}}\langle u^+_{p-q}u^+_{j-p}u^+_{k-j}\rangle + \cdots\Big\} \end{aligned} \tag{27}$$

$L_{03}\equiv \partial_t+\nu_0 p^2+\nu_0|\boldsymbol{j}-\boldsymbol{p}|^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2$, and where higher-order terms are easily found by induction. Thus, in all, equation (23) for the low-wavenumber modes may be written as

$$(\partial_t+\nu_0 k^2)u^-_k - \int_{-\infty}^{t}\mathrm{d}s\ A(\boldsymbol{k},t-s)u^-_\alpha(\boldsymbol{k},s) = M^-_k u^-_j u^-_{k-j} + S(k|K_c). \tag{28}$$

In order to test the hypothesis, we make two approximations. First, we truncate the expansion of $A(\boldsymbol{k},t)$ at lowest non-trivial order. This can be justified by the introduction of a *local* Reynolds number based on a length scale K_c^{-1}, the moment expansion being re-expressed as a power series in this parameter. Making the truncation in (26) and (27) leaves us with the expression

$$\begin{aligned} \lim_{\xi\to 0} M^-_{\alpha\beta\gamma}(\boldsymbol{k})\quad \langle u^+_\beta(\boldsymbol{j},t)u^+_\gamma(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c &= \int_{-\infty}^{t}\mathrm{d}s\ \exp[-(\nu_0 j^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2)(t-s)] \\ &\times 4M^-_{\alpha\beta\gamma}(\boldsymbol{k})M^+_{\beta\delta\epsilon}(\boldsymbol{j})\int \mathrm{d}^3 p\ \lim_{\{\cdot\}\to K_{\max}}\langle u^+_\epsilon(\boldsymbol{j}-\boldsymbol{p},s)u^+_\gamma(\boldsymbol{k}-\boldsymbol{j},s)\rangle u^-_\delta(\boldsymbol{k},s). \end{aligned} \tag{29}$$

For stationary, homogeneous, and isotropic turbulence we may write

$$\langle u^+_\epsilon(\boldsymbol{j}-\boldsymbol{p},s)u^+_\gamma(\boldsymbol{k}-\boldsymbol{j},s)\rangle = Q(|\boldsymbol{k}-\boldsymbol{j}|)D_{\epsilon\gamma}(\boldsymbol{k}-\boldsymbol{j})\delta(\boldsymbol{k}-\boldsymbol{p}) \tag{30}$$

where $Q(k)$ is the spectral density and δ is the Dirac delta function. This leaves the question of how to perform the time integral

$$\int_{-\infty}^{t}\mathrm{d}s\ \exp[-(\nu_0 j^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2)(t-s)]u^-_\delta(\boldsymbol{k},s). \tag{31}$$

To do this we change the variable of integration from s to $T=t-s$, expand the resultant $u^-_\delta(\boldsymbol{k},t-T)$ as a Taylor series about $T=0$, and then truncate the expansion at zero order, this approach being based upon the physical idea that the u^- modes are slowly evolving on timescales defined by the inverse of $\nu_0 j^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2$.

We have investigated the validity of these two approximations using results from direct numerical simulations performed on a 256^3 grid, with Taylor–Reynolds number $R_\lambda=190$. At this resolution the simulations have a very limited inertial range (see [5, 6]), but nevertheless they indicate that there is a range of K_c ($K_c\gtrsim 0.5K_{\max}$) where both approximations give rise to error terms of less than unity, and that the magnitude of these errors will decrease as we increase R_λ to the large values where we may reasonably expect our hypothesis to hold.

With these approximations, the right-hand side of (26) is simple to evaluate, and we are left with the final expression for the conditional average on the right-hand side of (23):

$$\begin{aligned} &M^-_{\alpha\beta\gamma}(\boldsymbol{k})\lim_{\xi\to 0}\langle u^+_\beta(\boldsymbol{j},t)u^+_\gamma(\boldsymbol{k}-\boldsymbol{j},t)\rangle_c \\ &\quad= 4M^-_{\alpha\beta\gamma}(\boldsymbol{k})M^+_{\beta\delta\epsilon}(\boldsymbol{j})\lim_{|\boldsymbol{k}-\boldsymbol{j}|\to K_{\max}}\frac{Q(|\boldsymbol{k}-\boldsymbol{j}|)D_{\epsilon\gamma}(\boldsymbol{k}-\boldsymbol{j})}{\nu_0 j^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2}u^-_\delta(\boldsymbol{k},t) \end{aligned} \tag{32}$$

which is linear in u_k^-, meaning that it may be interpreted in terms of an increment to the viscosity.

In order to evaluate the limit, we make a first-order truncation of a Taylor series expansion in wavenumber of Q^+ about $K_{\max}$. In this way, we re-obtain the results previously obtained using the two-field theory of McComb and Watt [7]. As they showed, a renormalization group calculation based upon these equations gives a prediction for the Kolmogorov constant of 1.60 ± 0.01, in good agreement with experiment, for $0.55K_{\max} \leqslant K_c \leqslant 0.75K_{\max}$. This calculation obtained the Kolmogorov exponent and pre-factor by assuming that the effective viscosity and its increment scale in the same way (which is true at the fixed point) and that the rate of energy transfer is renormalized. This latter assumption amounted, in our present terminology, to the neglect of $S(k|K_c)$ in equation (23).

A new justification of this step can now be offered as follows. The equation for the energy spectrum is obtained by multiplying the dynamical equation for $u_\alpha^-(\boldsymbol{k}, t)$ by $u_\alpha^-(-\boldsymbol{k}, t)$ and then performing an average over the full ensemble. Thus the effect of $S(k|K_c)$ is just

$$\langle S(k|K_c)u_\alpha^-(-\boldsymbol{k}, t)\rangle .$$

If we consider the form of $S(k|K_c)$ we see that each of the terms in the above expression involves a conditional average. In evaluating such terms we perform a double summation, firstly summing over all members with low-wavenumber modes close to a particular member of the ensemble, and then repeating this summation for every member of the ensemble. Now, the initial ensemble was constructed according to the principle of equal *a priori* probabilities but this is no longer necessarily true of the composite ensemble which we are now considering. If it were true, then the terms making up $S(k|K_c)$ would vanish identically for all K_c. However, in view of the results of the renormalization group calculations [7], it seems likely that the contribution from $S(k|K_c)$ is small for K_c in the range $0.55K_{\max} \leqslant K_c \leqslant 0.75K_{\max}$. Thus, for this range of cut-off wavenumbers, it would appear that the renormalization group calculation of the effective viscosity [7] is valid in a heuristic sense.

Finally, it should be noted that this work does not suggest that $S(k|K_c)$ can be neglected in equation (23), which is the governing equation for a single realization. However, it does suggest that, having averaged out the chaotic part to yield an effective viscosity, one should consider modelling the relationship of $S(k|K_c)$ to the u_k^- modes as predominantly deterministic. Work along these lines will be the subject of a separate communication.

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